Viability of complex self-interacting scalar field as dark matter

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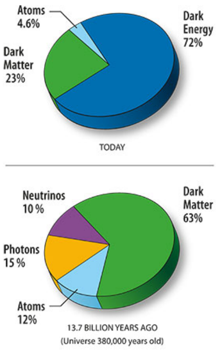
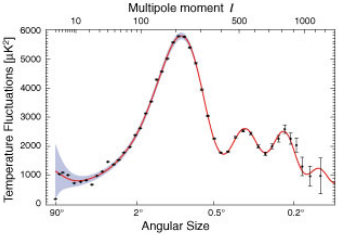
Dark matter at cosmological background level

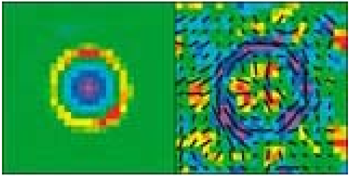
- Matter domination just before decoupling
- Dark matter particles may be relativistic at BBN and affect the effective number of extra neutrinos
- If very light, dark matter particles may affect the effective number of extra neutrinos at decoupling and at present time



Dark matter influence on CMB anisotropy

Dark matter influences BAO and therefore CMB temperature anisotropy in the peaks region





Dark matter influence on structure formation

Dark matter also influence structure formation in many ways

- During radiation domination δ growth is logarithmical
- During matter domination δ grows as a power of $a(t)$
- Depending on the EoS of dark matter and on the mass and interactions of dark matter particles, over-structures may occur. This gives bounds on the models.

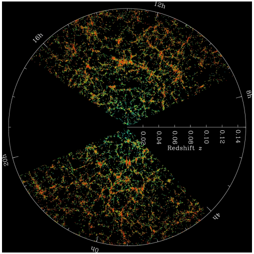


Figure: The SDSS galaxy map of the universe

Dark matter and galaxy rotation curves

The velocity of stars in circular orbit in the galaxy can be expressed as

$$v_c^2 = \frac{G\,M(r)}{r}, \qquad M(r) = \int_0^r r'^2\,\rho_m(r')\,dr'$$

This also gives bounds on the model

$$v_c \longrightarrow \rho_m \longrightarrow \text{model}$$

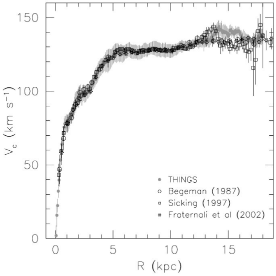


Figure: Comparison of the NGC 2403 rotation curve determinations from the literature.

Dark matter models

- WIMPS
- Axions
- f(R) modified gravity
- Mond
- SFDM (include B-E condensates)

In particular, a real non-self-interacting scalar field Bose-Einstein condensate requires

- An extremely low mass mass $\sim 10^{-23}\, eV$ to explain
  the observed amount of structure, the critical mass, the central density and the rotation curves of galaxies
- A condensation temperature $T_c \sim 10^{8}\, eV$
- A temperature $T \sim 10^{-27}\, eV$ today

The model

Let us consider a complex self interacting scalar field χ with renormalizable self-interacting potential

$$v(\chi, \bar{\chi}) = m_0^{\chi 2} |\chi|^2/2 + h |\chi|^4$$

as dark matter field. The presence of the self-interaction has many important consequences for the model.

- It allows the formation of a χ particles condensate at early times. The equilibrium configuration is that of a condensate in equilibrium with a thermalized gas of χ and $\bar{\chi}$ particles with temperature T_χ
- It gives a thermal correction to m_0^χ.
  At sufficiently high temperatures one has $m_{th}^\chi \sim T_\chi$ and $\rho_c^\chi \sim T_\chi^4$ scales as radiation. This help to solve the cosmological coincidence problem with less stringent constrains on the parameters
- It changes the spherically symmetric solutions, allowing masses of the order of $1 - 10^{-3}\, eV$.
  This value should be compared with the value $10^{-23}\, eV$ found in the non interacting case $h = 0$.

Constrains on the model

To constrain the model one use

- Effective number of extra neutrinos at BBN $\Delta_\nu^{eff} = 0.054^{+1.4}_{-1.2}$
- Effective number of extra neutrinos at CMB $\Delta_\nu^{eff} = 1.30^{+0.86}_{-0.88}$.
  This is less stringent than the BBN constrain
- Dark Matter Halos of a size of about 100 *Kpc*

Formalism

- Take $L = \frac{1}{2}\partial_\mu \chi \partial^\mu \chi - \frac{1}{2} m_0^\chi \left|\chi\right|^2 - h \left|\chi\right|^4$.
- Take $h \ll 1$ and consider interactions as perturbations
- Define the number densities of particles and anti-particles in terms of the phase space distributions f_χ and $f_{\bar{\chi}}$ as

$$n^\chi = \int \frac{d^3 p}{(2\pi)^3} f_\chi(p), \qquad n^{\bar{\chi}} = \int \frac{d^3 p}{(2\pi)^3} f_{\bar{\chi}}(p)$$

- The energy density and the charge density of the complex χ field are

$$\rho^\chi = \int \frac{d^3 p}{(2\pi)^3} E_\chi(p) \left[f_\chi(p) + f_{\bar{\chi}}(p)\right], \qquad Q^\chi = \int \frac{d^3 p}{(2\pi)^3} \left[f_\chi(p) - f_{\bar{\chi}}(p)\right]$$

The equilibrium configuration with the condensate

The equilibrium configuration is

$$f_\chi(p) = f_\chi^{BE}(p) + (2\pi)^3\, Q_c\, \delta^3(p), \qquad f_{\bar\chi}(p) = f_{\bar\chi}^{BE}(p)$$

where

- $f_\chi^{BE}(p) = \frac{1}{\left[e^{\beta\,(E-\mu)}-1\right]}$, is the Boltzmann distribution of χ particles
- $f_{\bar\chi}^{BE}(p) = \frac{1}{\left[e^{\beta\,(E+\mu)}-1\right]}$ is the Boltzmann distribution of χ anti-particles
- μ is the chemical potential.
- Q_c is the density number of the χ particles of the condensate.

Thermal corrections to the scalar field mass

At very high temperatures one expect the thermal corrections to m_0^χ to be dominant and m_0^χ to be negligible, so

$$(m_{th}^\chi)^2 \simeq 4h \int \frac{d^3p}{(2\pi)^3\, 2E}\,(f_\chi(p) + f_{\bar\chi}(p)) \simeq \left(2hQ_c + \frac{1}{3} h T_\chi^3\right) / m_{th}^\chi$$

Therefore, at high temperatures one has

$$m_{th}^\chi \simeq \alpha \cdot\ T_\chi \text{ with } \alpha \equiv \left[h\left(2\frac{Q_c}{T_\chi^3} + \frac{1}{3}\right)\right]^{1/3}$$

In conclusion the χ mass will be given by

$$m^\chi \simeq m_0^\chi \qquad \textit{for} \qquad T_\chi \leq \frac{m_0^\chi}{\alpha} \equiv T_1^\chi$$

$$m^\chi \simeq m_{th}^\chi(Q_c, T_\chi, h) \qquad \textit{for} \qquad T_\chi \gg \frac{m_0^\chi}{\alpha} \equiv T_1^\chi$$

- The total χ number density is $n^{\chi} \simeq Q_c + n^{\chi}_{th}$
  where $n^{\chi}_{th} \equiv \int \frac{d^3p}{(2\pi)^3} f^{BE}_{\chi}(p)$ is the number density of thermaized χ
- The number density of $\bar{\chi}$ is $n^{\bar{\chi}} = n^{\bar{\chi}}_{th} \equiv \int \frac{d^3p}{(2\pi)^3} f^{BE}_{\bar{\chi}}(p)$
- The energy density is
  $\rho^{\chi} \simeq m^{\chi} Q_c + \rho^{\chi}_{th}$
  where $\rho^{\chi}_{th} = \int \frac{d^3p}{(2\pi)^3} E_{\chi}(p) \left[f^{BE}_{\chi}(p) + f^{BE}_{\bar{\chi}}(p) \right]$ is the energy density of particles in thermal configuration
- The charge density is $Q^{\chi} \simeq Q_c + Q^{\chi}_{th}$
  where $Q^{\chi}_{th} = n^{\chi}_{th} - n^{\bar{\chi}}_{th} = \int \frac{d^3p}{(2\pi)^3} \left[f^{BE}_{\chi}(p) - f^{BE}_{\bar{\chi}}(p) \right]$

Note that, since $m^{\chi}_{th}(Q_c, T_{\chi}, h)$ depends on Q_c, T_{χ} and h, also n^{χ}_{th}, ρ^{χ}_{th} and Q^{χ}_{th} in general depends on Q_c, T_{χ} and h through the χ mass.
Anyhow at $T_{\chi} \gg m^{\chi} > \mu$ one can neglect m^{χ} and μ and recover the usual result

$$n^{\chi}_{th} = n^{\bar{\chi}}_{th} = \frac{\zeta(3)}{\pi^2} T^3_{\chi}, \quad \rho^{\chi}_{th} = \frac{\pi^2}{15} T^4_{\chi}, \quad Q^{\chi}_{th} = \frac{\mu(T_{\chi})}{3} T^2_{\chi}$$

Cosmological evolution of T_χ and Q_c

For any initial value of Q_c/T_χ^3, the equilibrium configuration

$$f_\chi(p) = f_\chi^{BE}(p) + (2\pi)^3\, Q_c\, \delta^3(p), \qquad f_{\bar\chi}(p) = f_{\bar\chi}^{BE}(p)$$

is solution of the relativistic Boltzmann equation for

- $T_\chi \sim 1/a(t)$
- $Q_c \sim T_\chi^3 \sim 1/a(t)^3$
- $\frac{m-\mu}{T_\chi} \ll 1$

at any temperature $T_\chi \gg m^\chi \geq \mu > 0$ and one has

- Q_c/T_χ^3 is constant as long as $T_\chi \gg m^\chi$
- $\alpha \equiv \left[h\left(2\frac{Q_c}{T_\chi^3} + \frac{1}{3}\right)\right]^{1/3}$ is constant as long as $T_\chi \gg m^\chi$ and therefore $m^\chi_{th} = \alpha \cdot T_\chi \propto T_\chi$.
- The condition $T_\chi \gg m^\chi$ implies $\alpha \ll 1$
- $k = \frac{T_\chi}{T_\gamma}$ is also constant

Cosmological evolution

Define the following temperatures

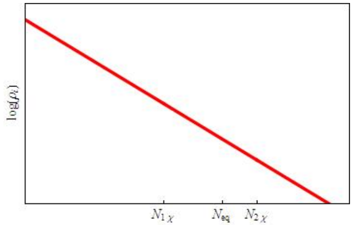
- $T_{1\chi} \equiv m_0^\chi/\alpha, \quad T_{2\chi} \equiv m_0^\chi$
- The corresponding photons temperatures are
  $T_{1\gamma} \equiv T_{1\chi}/k, \quad T_{2\gamma} \equiv T_{2\chi}/k$

One can divide the scalar field evolution in three cosmological epochs

- $T_\chi \gg T_{1\chi}$ where $m^\chi \simeq m_{th}^\chi = \alpha \cdot T_\chi$ and $T_\chi \gg m^\chi$
  therefore $\rho_c^\chi \simeq m_{th}^\chi Q_c \sim T_\chi^4$ and $\rho_{th}^\chi \sim T_\chi^4$
- $T_{1\chi} \gg T_\chi \gg T_{2\chi}$ where $m^\chi \simeq m_0^\chi$
  $\rho_c^\chi \simeq m_0^\chi Q_c \sim T_\chi^3$ and $\rho_{th}^\chi \sim T_\chi^4$
  The density parameter of the condensate $\Omega_c \equiv \rho_c^\chi/3\rho_{tot}$ begins to grow since $\rho_c^\chi/\rho_{rad}^\chi \sim a(t)$
- $T_\chi \ll T_{2\chi}$
  The thermalized χ particles are non relativistic so the whole condensate energy density behave as matter $\rho^\chi \sim a(t)^3$

Remarks

- The thermalized $\chi - \bar{\chi}$ gas is relativistic at temperatures $T_\chi \gg T_{1\chi} \equiv m_o^\chi$ i.e. m_0^χ fixes the temperature at which the termalized gas becomes non-relativistic
- The condensate evolves as radiation at temperatures $T_\chi \gg T_{1\chi} \equiv m_o^\chi/\alpha$ i.e. m_0^χ and α fixe the temperature at which the condensate becomes non-relativistic
- One has two parameters, m_0^χ and α that fix $T_{1\chi}$ and determine the begin matter domination
- at temperatures $T_\chi < T_{1\chi}$, the DM density parameter Ω_c begins to grow Radiation-Matter equality is then fixed by ρ_c, m_0^χ and α: no fine tuning
- In the case without self-interaction the condensate always evolves as matter: **fine tuning on ρ_c^χ !!!**



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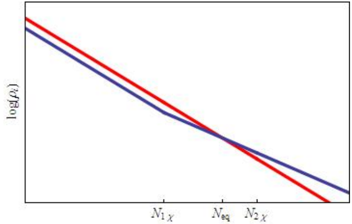

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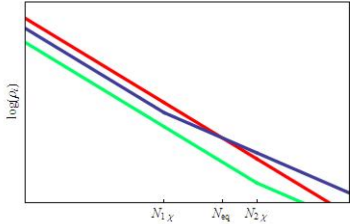

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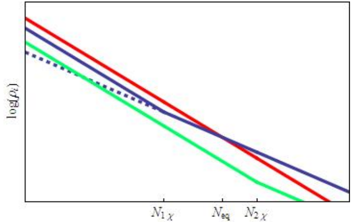

Condensate formation

- The χ field is produced via inflaton decay with a charge asymmetry such that $Q^\chi > 0$, via an Affleck-Dine mechanism
  M. Dine, Nucl. Phys. B249, 361 (1985).
- Just one has $\rho^\chi \propto a^{-4}$ and $R \equiv Q^\chi / \rho^{\chi 3/4}$ is constant
- The condition for the condensate formation is $R \geq 0.2\, h^{1/2}$
  G.Mangano, G.Miele,S.Pastor,M. Peloso. Phys.Rev.D64:123509,2001
- This is re expressed as $R = \dfrac{\frac{Q_c}{T_\chi^3} + \frac{\mu(T_\chi)}{3T_\chi}}{\left(\frac{\alpha\, Q_c}{T_\chi^3} + \frac{\pi^2}{15}\right)^{3/4}} > 0.2\, h^{1/2}$

  This condition should be verified at posteriori after the determination of the parameters h and m_0^χ and initial condition for Q_c / T_χ^3

Effective number of extra neutrinos

- As long as $T_\chi \gg m_0^\chi$, the thermalized $\chi - \bar{\chi}$ gas behaves as radiation. The number of extra neutrinos related to it is $\Delta_\nu^{th} = \frac{16}{7}\left(\frac{T_\chi}{T_\nu}\right)^4$
- As long as $T_\chi \gg m_0^\chi/\alpha$, the condensate behaves as radiation. The number of extra neutrinos related to it is $\Delta_\nu^{c} = \frac{240}{7\pi^2}\left(\frac{T_\chi}{T_\nu}\right)^4 \frac{\alpha\, Q_c}{T_\chi^3}$

These should be in the BBN and CMB limits in the range of temperatures where the condensate and the thermalized gas are relativistic

- BBN bound$\Delta_\nu^{eff} = 0.054^{+1.4}_{-1.2}$
- CMB bound $\Delta_\nu^{eff} = 1.30^{+0.86}_{-0.88}$

Dark matter halos

Taking $\chi = \sigma(r) e^{i\omega t}/\sqrt{2}$ the field equations in newtonian limit read

$$\triangle \Phi = 4\pi G \left(2\omega^2 \sigma^2 - \frac{1}{2} m_0^{\chi 2} \sigma^2 - \frac{1}{2} h \sigma^2 \right)$$

$$\triangle \sigma + (1 - 4\Phi)\, \omega^2 \sigma - (1 - 2\Phi) \left(\frac{m_0^{\chi 2}}{2} \sigma + h \sigma^3 \right) = 0$$

For $\Lambda \equiv h / \left(4\pi\, G\, m_0^{\chi 2}\right) \gg 1$, $\Lambda^{-1} \ll \sigma_0 / M_p \ll \Lambda^{-1/2}$ and $r \leq \pi\, L_H$ one has (A.Arbey, J.Lesgourgues, P.Salati. Phys.Rev.D65:083514,2002)

$$\sigma(r) = \sigma_0 \sqrt{\frac{sin(r/L_H)}{(r/L_H)}}, \qquad r\phi'(r) = v(r) = 2\pi \Lambda \frac{\sigma_0^2}{M_p^2} \left[\frac{sin(r/L_H)}{(r/L_H)} - cos(r/L_H) \right]$$

$$L_H \equiv h^{1/2} \frac{M_p}{m_0^{\chi 2}}, \qquad \text{or} \qquad m_0^{\chi} = 3\, h^{1/4} \sqrt{\frac{Kpc}{L_H}}\, eV$$

We only know that the size of the halos is at least $\pi\, L_H$ so:
L_H is a minimum for the size of DM halos and any value $L_H \leq 100\, Kpc$ is viable

Is the model viable?

In A.Arbey, J.Lesgourgues, P.Salati. Phys.Rev.D65:083514,2002. e-Print: astro-ph/0112324, it is argued that choosing $L_H \simeq 10\,Kpc$ (the dimension of the core od DM halos) one obtains $\Delta_\nu^{eff} \simeq 5$ and they concludes that the model is not viable.
This is correct?

- The analytical solution is incomplete, since it is valid for $r \leq \pi\, L_H$.
  No indication about the behavior of the solution elsewhere.
- The halo could be much greater than L_H for many orders of magnitude.
- L_H is a lower bound for the size od DM halos, therefore any $L_H \leq 100\,Kpc$ is viable.
- Since $\Delta_\nu^c \simeq 3.34 \left(\frac{L_H}{Mpc}\right)^{2/3}$, lowering L_H one obtains a viable Δ_ν^{eff}.
- The analysis of spherical solutions is incomplete, since it does not account for the contribution of baryons, radiation and thermalized $\chi - \bar{\chi}$ gas.

Bounds on the thermalized $\chi - \bar{\chi}$ gas

- At BBN it should be $\Delta_\nu^{th} = \frac{16}{7}\left(\frac{T_\chi}{T_\nu}\right)^4 \leq 0.054^{+1.4}_{-1.2}$
- This gives $k \equiv T_\chi / T\gamma \leq 0.8$
- One has $\rho_{th}^\chi / \rho_{rel} \leq k^4 / g_r \leq 0.3 \cdot k^4$, for $g_r \geq 3.36$
- Therefore at matter domination one will have $\rho_{th}^\chi < \rho_{rel} \ll \rho_c^\chi$
- At radiation-matter equality the condensate should evolve as matter so $T_\chi^{eq} = k\, T_\gamma^{eq} \ll T_{1\chi}$
- Therefore we should require

  $T_\gamma^{eq} \simeq 0.698\, eV \ll \frac{m_0^\chi}{k\alpha}$

Bounds on the condensate

Introduce a parameter n as $L_H \simeq \frac{0.1}{n^2}\, Kpc \Longrightarrow \qquad m_0^\chi \simeq 10\, n\, h^{1/4}$

- t radiation matter equality $\rho_\chi^{c\; eq} = m_0^\chi\, Q_c^{eq} = \rho_{DM}^{eq} \simeq 0.323\, eV^4$
- $\frac{Q_c}{T_\chi^3} = \frac{Q_c^{eq}}{T_\chi^{eq3}} = \frac{9.5 \cdot 10^{-2}}{n\, k^3 h^{1/4}}$
- For $Q_c/T_\chi^3 \geq 1$ one has $\alpha \simeq \frac{0.57\, h^{1/4}}{n^{1/3} k}$
- One has $\Delta_\nu^c \simeq \frac{0.72}{n^{4/3}}$
  Only depends on n
- From BBN bounds one has $n \geq 0.78$

Temperatures

Using $m_0^\chi \simeq 10\, n\, h^{1/4}$ and $\alpha \simeq \frac{0.57\, h^{1/4}}{n^{1/3} k}$ one has

$$T_{1\chi} \simeq 17.6\, k\, n^{4/3}\, eV, \qquad T_{2\chi} \simeq 10\, n\, h^{1/4}\, eV, \qquad T_\chi^{eq} = 0.69\, k\, eV$$

- The condition $T_{1\chi} \geq T_\chi^{eq} = k\, T_\gamma^{eq}$ gives $n \geq 0.088$
- The condition $T_{1\chi} \gg T_{2\chi}$ implies that $h \ll 10\, k^4\, n^{4/3}$

Realistic model

$$\Delta_\nu^c \simeq 3.34 \left(\frac{L_H}{Mpc}\right)^{2/3}, \qquad L_H \simeq \frac{0.1}{n^2} Mpc$$

Take $n \simeq 0.8$ and $k \simeq 0.3$

- $L_H \simeq 0.17\, Kpc$ that is well below the typical size for DM halos
- $Q_c/T_\chi^3 \sim 10\, h^{-1/4} \geq 1$ for any $h < 1$ and $\alpha \simeq 2\, h^{1/4}$
- $R \simeq h^{-1/4} \geq 0.2\, h^{1/2}$, so the condensate forms at early times.
- $\Delta_\nu^c \simeq 0.97$
- $m_0^\chi \simeq 8\, h^{1/4}\, eV$ much greater than the value $m \simeq 10^{-27}\, eV$ in non self-interacting case
- The condition $T_{1\chi} \geq T_\chi^{eq} = k\, T_\gamma^{eq}$ is verified since $n \geq 0.088$
- The condition $T_{1\chi} \gg T_{2\chi}$ implies that $h \ll 10^{-2}$

t	t_1	t_2	t_{eq}
T_χ	$0.39\, eV$	$8\, h^{1/4}\, eV$	$0.21\, eV$
T_γ	$13\, eV$	$26.7\, h^{1/4}\, eV$	$0.698\, eV$

h	10^{-4}	10^{-8}	10^{-12}
α	0.2	0.02	0.002
m_0^χ	$0.8\, eV$	$0.08\, eV$	$0.008\, eV$
$T_{2\gamma}$	$2.67\, eV$	$0.267\, eV$	$0.002\, eV$

Conclusions

Advantages of the model

- No fine tuning on ρ_χ at early times
- Mechanism for the B-E condensate formation
- Realistic value of the mass as large as 1 eV

Further studies

- Growth of anisotropy in the linear regime via the Boltzmann equation
- Comparison with WMAP and Planck data
- Structure formation in the non linear regime

Comments on Harko, arXiv:1101.3655 [gr-qc]

The author use the Gross-Pitaevskii equation

$$i\hbar\partial_t\psi(\vec{r},t) = \left[-\hbar^2\nabla^2 + V_{rot}(r) + V_{ext}(r) + g'(|\psi(\vec{r},t)|^2)\right]\psi(\vec{r},t)$$

with

$$g(\rho_m) = \tfrac{u_0}{2}|\psi|^4 = \tfrac{u_0}{2}|\rho_m|^2$$

$$P_m(\rho_m) = U_0\rho_m^2 \tag{1}$$

$$\omega_{eff} = P_m/\rho_m = U_0\rho_m$$

and, in the hydrodynamical representation, he obtains the solution

$$\rho_b(a) = \tfrac{c^2}{U_0}\tfrac{\rho_0}{(a/a_0)^3 - \rho_0} \tag{2}$$

- This solution is singular at $(a/a_0)^3 = \rho_0$, not at the same time of the metric and other energy densities that are singular at $a = 0$.
- Problems with radiation domination

$$\Omega_r = \Omega_r^0 \, \frac{1}{(a/a_0)^4}, \qquad \Omega_{BE} = \Omega_{DM}^0 \, \frac{1-\rho_0}{(a/a_0)^3-\rho_0} \tag{3}$$

with $\Omega_{DM}^0 \gg \Omega_r^0$. Therefore the ratio between the two density parameters is

$$\begin{aligned} \Omega_{BE}/\Omega_r &= (\Omega_{DM}^0/\Omega_r^0)\,(a/a_0)\,\frac{1-\rho_0}{1-\rho_0/(a/a_0)^3} = (\Omega_{DM}^0/\Omega_r^0)\,f(a) \\ f(a) &= (a/a_0)\,\frac{1-\rho_0}{1-\rho_0/(a/a_0)^3} \end{aligned} \tag{4}$$

The minimum of Ω_{BE}/Ω_r is at
$f'(a) = (1/a_0)\,\frac{1-\rho_0}{(1-\rho_0/(a/a_0)^3)^2}\left[1 - 4\rho_0/(a/a_0)^3\right] = 0$ i.e. at $a_{min} = a_0\,(4\,\rho_0)^{1/3}$
where

$$(\Omega_{BE}/\Omega_r)_{min} = (\Omega_{DM}^0/\Omega_r^0)\,\frac{4}{3}\,(4\,\rho_0)^{1/3}\,(1-\rho_0) \gg 1 \tag{5}$$

since $\Omega_{DM}^0/\Omega_r^0 \gg 1$ and $\rho_0 \simeq 0.7$ in the model