The (2+1) Dimensional Black Hole with Scale Dependence

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- BTZ with Running Couplings
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Introduction

The understanding of quantum gravity allow us to get insights in the Black Hole theory.

At low energies, the resulting effective action of gravity show us a **scale dependence**.

Thus, the couplings appearing in the quantum-effective action as scale dependent quantities, such as $G_0 \to G_k$ and $\Lambda_0 \to \Lambda_k$.

Classical BTZ Solution

The gravitational action in three dimensions is

$$\mathcal{S}(g_{\mu\nu}) = \int d^3x\sqrt{-g}\,\frac{(R-2\Lambda_0)}{G_0},$$

which leads to

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\Lambda_0 g_{\mu\nu},$$

being Λ_0 and G_0 the cosmological constant and the Newton's constant respectively.

The line element for a non rotating black hole in (2+1) looks like:

$$ds^2 = -f_0(r)\,dt^2 + f_0(r)^{-1}\,dr^2 + r^2 d\phi^2,$$

Classical BTZ Solution

With the solution given by

$$f_0(r) = -G_0 M_0 + \frac{r^2}{\ell_0^2}.$$

The entropy and temperature is:

$$S_0 = 4\pi\ell_0\sqrt{\frac{M_0}{G_0}}\,, \qquad\qquad T_0 = \frac{\sqrt{M_0 G_0}}{2\pi\ell_0}.$$

where $\Lambda_0 \equiv -1/\ell_0^2$ and M_0 is the mass of Black Hole.

Please, note that the horizon is given by the condition $f_0(r_H) = 0$

$$r_H = \pm\sqrt{G_0 M_0}\,\ell_0$$

BTZ with Running Couplings

The gravitational action in three dimensions is

$$\Gamma(g_{\mu\nu}, k) = \int d^3x \sqrt{-g} \frac{(R - 2\Lambda_k)}{G_k}.$$

Thus, varying with respect to the metric field

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\Lambda_k g_{\mu\nu} + 8\pi G_k T_{\mu\nu},$$

where the **effective** energy-momentum tensor is given by

$$8\pi G_k T_{\mu\nu} = 8\pi G_k T^m_{\mu\nu} - \Delta t_{\mu\nu}$$

being the new term:

$$\Delta t_{\mu\nu} = G_k \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) G_k^{-1}$$

BTZ with Running Couplings

In the same way, by varying the action with respect to the scale-field $k(x)$ one obtains the algebraic equations

$$\left[R\frac{\partial}{\partial k}\left(\frac{1}{G_k}\right) - 2\frac{\partial}{\partial k}\left(\frac{\Lambda_k}{G_k}\right)\right] = 0$$

However, we don't use it!

On the other hand, we take into account that:

$$\mathcal{O}(k) \longrightarrow \mathcal{O}(k(r)) \longrightarrow \mathcal{O}(r)$$

And solve with respect to the radial variable.

BTZ with **Running Couplings**

The metric has a general form:

$$ds^2 = -f(r)\,dt^2 + g(r)\,dr^2 + r^2 d\phi^2.$$

Note that, in order to combine $f(r)$ with $g(r)$, we will use the so called "**null energy condition**":

$$T_{\mu\nu} l^\mu l^\nu \stackrel{!}{=} T^m_{\mu\nu} l^\mu l^\nu \geq 0.$$

Which produces:

$$R_{\mu\nu} l^\mu l^\nu = (f \cdot g)' \frac{1}{2rg} = 0,$$

which allow us to rewrite

$$f(r) \cdot g(r) = 1.$$

BTZ with Running Couplings

Thus, the running couplings let us to get the Newton's function:

$$G(r) = a \left[\int_{r_0}^{r} \sqrt{f(r') \cdot g(r')} \, dr' \right]^{-1},$$

with the general solution:

$$G(r) = \frac{G_0}{1 + \alpha r}$$

Note that the functional form of $G(r)$ is a consequence of the null energy condition!

Generalized Solution

Thus, the solution for Einstein field equations, considering the null energy condition and the equation $\delta\mathcal{S}/\delta k$ is:

$$G(r) = \frac{G_0^2}{G_0 + \epsilon r(1 + G_0 M_0)},$$

$$f(r) = f_0(r) + 2M_0 G_0 \left(\frac{G_0}{G(r)} - 1\right)\left[1 + \left(\frac{G_0}{G(r)} - 1\right)\ln\left(1 - \frac{G(r)}{G_0}\right)\right],$$

$$\Lambda(r) = \frac{-G(r)^2}{\ell_0^2 G_0^2}\left[1 + 4\left(\frac{G_0}{G(r)} - 1\right) + \left(5M_0 G_0 \frac{\ell_0^2}{r^2} + 3\right)\left(\frac{G_0}{G(r)} - 1\right)^2 + 6M_0 G_0 \frac{\ell_0^2}{r^2}\left(\frac{G_0}{G(r)} - 1\right)^3\right.$$
$$\left. + 2M_0 G_0 \frac{\ell_0^2}{r^2}\frac{G_0}{G(r)}\left(3\left(\frac{G_0}{G(r)} - 1\right) + 1\right)\left(\frac{G_0}{G(r)} - 1\right)^2 \ln\left(1 - \frac{G(r)}{G_0}\right)\right]$$

Classical Limits

Note that ϵ parametrizes the strength of the scale dependence and when $\epsilon \to 0$ the classical solution is recovered.

$$\lim_{\epsilon \to 0} G(r) = G_0 \quad , \quad \lim_{\epsilon \to 0} f(r) = -G_0 M_0 + \frac{r^2}{\ell_0^2} \quad , \quad \lim_{\epsilon \to 0} \Lambda(r) = -\frac{1}{\ell_0^2}$$

And the mass gap is given by

$$\lim_{\substack{\epsilon \to 0 \\ M_0 \to -1/G_0}} G(r) = G_0 \quad , \quad \lim_{\substack{\epsilon \to 0 \\ M_0 \to -1/G_0}} f(r) = 1 + \frac{r^2}{\ell_0^2} \quad , \quad \lim_{\substack{\epsilon \to 0 \\ M_0 \to -1/G_0}} \Lambda(r) = -\frac{1}{\ell_0^2}$$

Note further that M_0 is the mass of the black hole only if $\epsilon \to 0$, while for $\epsilon \neq 0$ it is much harder to determine the mass.

Asymptotic space-times

For small radial coordinate a new singularity appears, which is absent in the classical BTZ solution:

$$R = -f''(r) - 2\frac{f'(r)}{r},$$

Due lapse function has a lineal term, the singularity can not be avoided. In fact, since ϵ controls the strength of the new scale dependence effects, it is useful to treat it as small expansion parameter:

$$G(r) = G_0 - \epsilon \cdot (1 + G_0 M_0) r + \mathcal{O}(\epsilon^2)$$

$$f(r) = -G_0 M_0 + \frac{r^2}{\ell_0^2} + 2\epsilon \cdot M_0 (1 + G_0 M_0) r + \mathcal{O}(\epsilon^2),$$

$$\Lambda(r) = -\frac{1}{\ell_0^2} - \epsilon \cdot \frac{2r}{\ell_0^2 G_0}(1 + G_0 M_0) + \mathcal{O}(\epsilon^2).$$

Asymptotic space-times

The reason is that only a very particular class of lapse functions renders the Ricci scalar finite at the origin.

This can be seen by solving R for finite and constant $R = b$ at $r \to 0$. The solution to this is

$$f(r) \approx A + \frac{B}{r} + Cr^2,$$

This shows that any lapse function which has a linear term in r (or any other power r^n with $n \leq 2$ and $n \neq \{-1, 0\}$) necessarily produces a divergence in the Ricci scalar at $r \to 0$.

Horizon structure

The zero of the lapse function implies a transcendental equation for radial coordinate, which **can not be solved analytically.**

We approach this problem in three different ways:

1- Expansion in $\epsilon \ll 1$:

$$r_H = \sqrt{G_0 M_0} \ell_0 - \epsilon \ell_0^2 M_0 (1 + G_0 M_0) + \mathcal{O}(\epsilon^2).$$

Unfortunately, an analytic result is again limited to order ϵ . One sees that the scale dependence tends to decrease the apparent horizon radius.

2a- Expansion in $G(r_H)/G_0 \ll 1$:

Newton's coupling will be much smaller than its classical value:

$$\epsilon r_H (1 + M_0 G_0) \gg G_0$$

Horizon structure

In this limit we have

$$r_H^3 \approx \frac{2}{3}\frac{M_0 G_0^2 \ell_0^2}{(1+M_0G_0)\epsilon}.$$

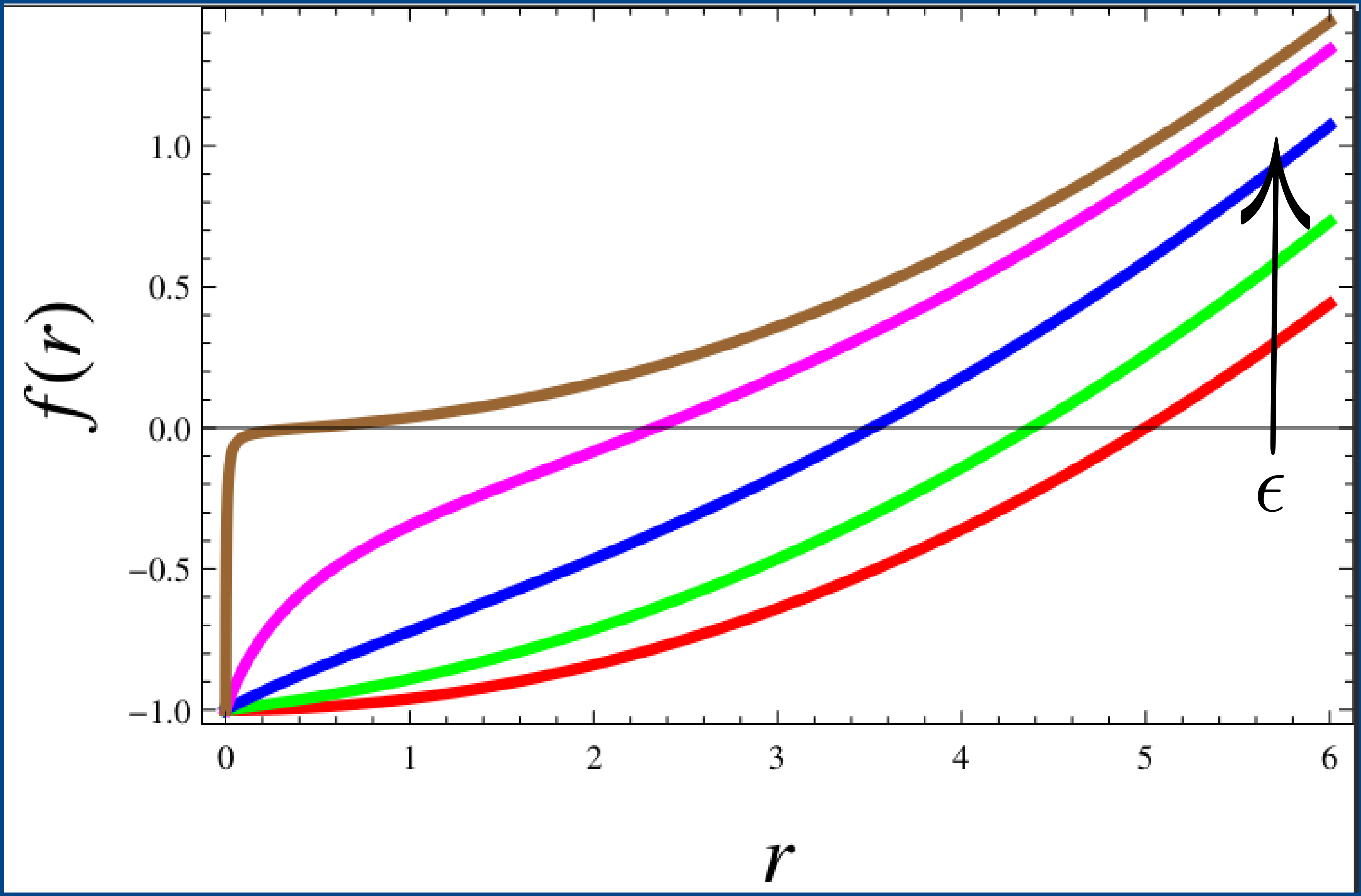
2b- Expansion in $G(r_H)/G_0 \ll 1$ and $G_0 M_0 \gg 1$:

$$r_H^3 \approx \frac{2}{3}\frac{G_0 \ell_0^2}{\epsilon}.$$

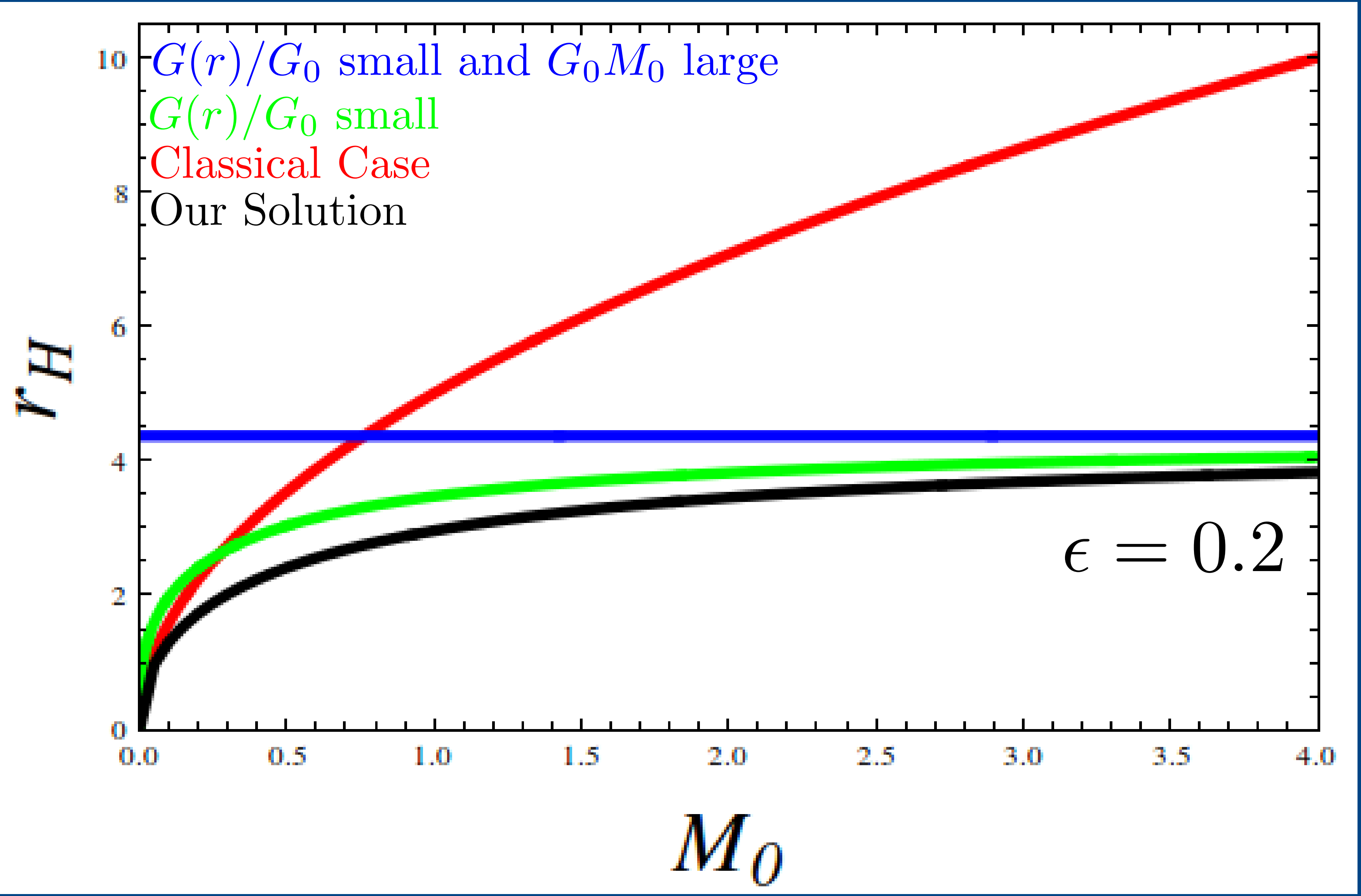
We see the crucial difference with its constant-coupling counterpart: for a fixed cosmological constant, the radius of the horizon remains finite as $M_0 \to \infty$.

3- Numerical analysis

Numerical Results A



Numerical Results B



Black Hole Thermodynamics

We found the black hole temperature and the entropy by use the standard relations:

$$S = \frac{A}{4G(r)}\bigg|_{r_H}, \qquad T = \frac{1}{4\pi}\frac{df(r)}{dr}\bigg|_{r_H}.$$

Where corrections in both quantities are induced by scale dependence framework.

$$T = \frac{1}{2\pi r}\frac{G_0^2 M_0}{G_0 + r\epsilon(1 + G_0 M_0)}\bigg|_{r_H}.$$

$$S = \frac{A}{4G_0}\left[1 + \frac{(1 + G_0 M_0)\epsilon r}{G_0}\right]\bigg|_{r_H}.$$

Black Hole Thermodynamics

Temperature and entropy admit approximations such as:

Temperature

1- Expansion in $\epsilon \ll 1$ **:**

$$T = \frac{\sqrt{M_0 G_0}}{2\pi \ell_0} + O(\epsilon^2).$$

2a- Expansion in $G(r_H)/G_0 \ll 1$ **:**

$$T \approx \frac{1}{4\pi} \left(18 \frac{M_0 G_0^2}{\ell_0^4 (1 + G_0 M_0)\epsilon} \right)^{1/3},$$

Black Hole Thermodynamics

2b- Expansion in $G(r_H)/G_0 \ll 1$ and $G_0 M_0 \gg 1$ **:**

$$T| \approx \frac{1}{4\pi}\left(18\frac{G_0}{\ell_0^4 \epsilon}\right)^{1/3} .$$

Entropy

1- Expansion in $\epsilon \ll 1$ **:**

$$S = \frac{\pi}{2}\sqrt{\frac{M_0}{G_0}}\ell_0 + \mathcal{O}(\epsilon^2).$$

2a- Expansion in $G(r_H)/G_0 \ll 1$ **:**

$$S \approx \pi\left[\frac{\ell_0^4 M_0^2 (1 + M_0 G_0)\epsilon}{18 G_0^2}\right]^{1/3} .$$

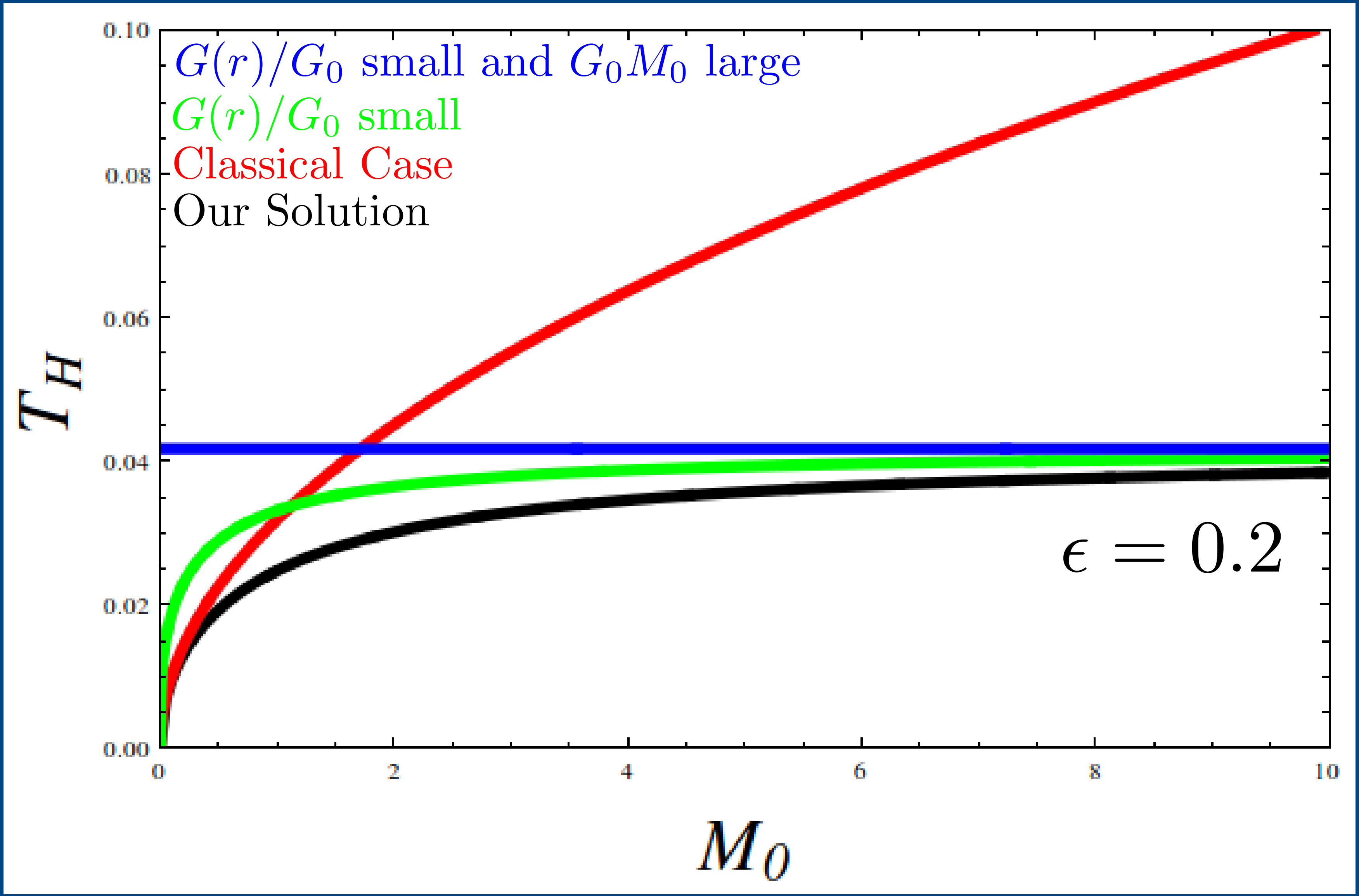
Black Hole Thermodynamics

2b- Expansion in $G(r_H)/G_0 \ll 1$ and $G_0 M_0 \gg 1$ **:**

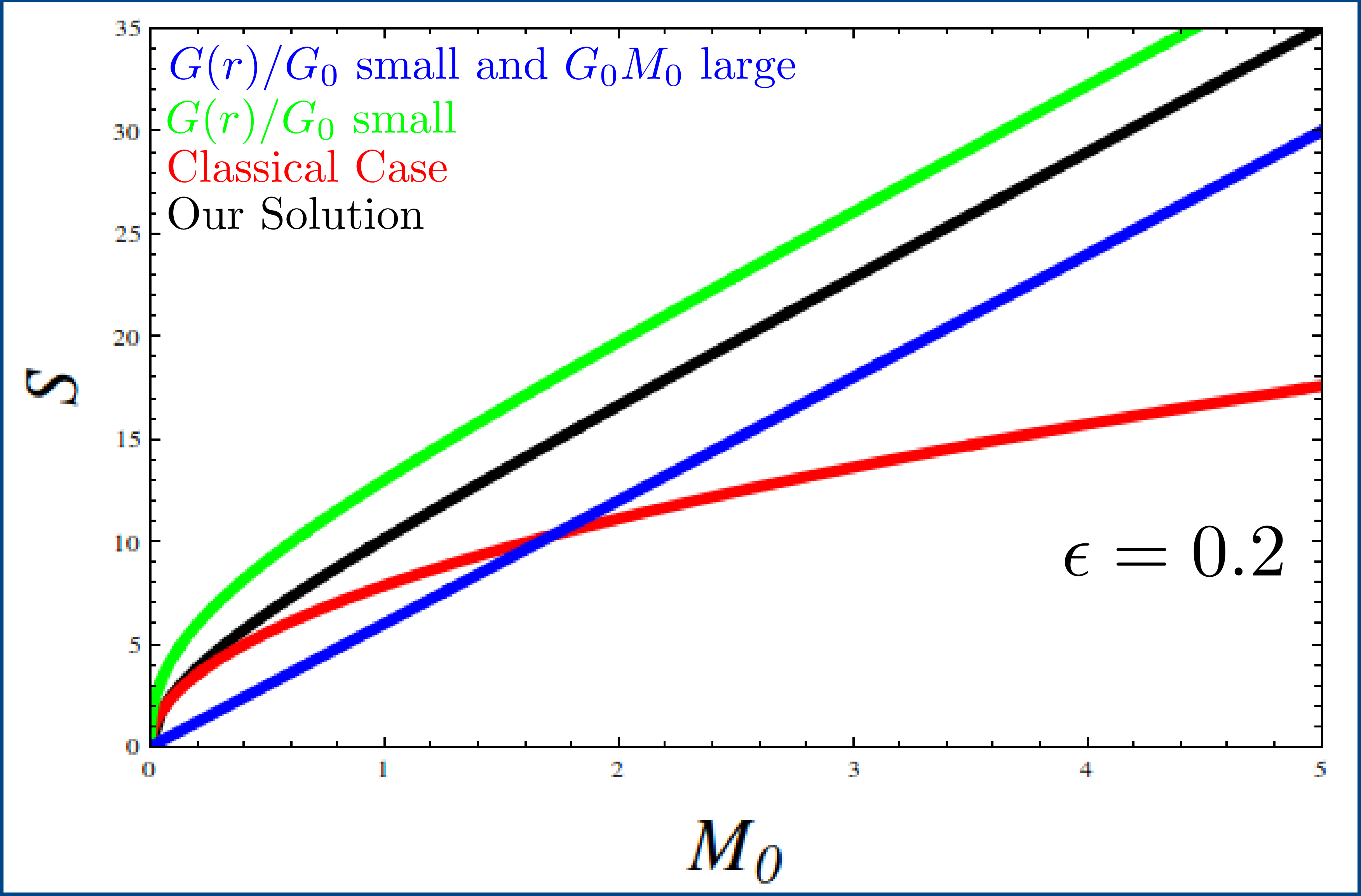
$$S \approx \frac{A}{4G_0} M_0 \epsilon r_h = \pi M_0 \left(\frac{\ell_0^4 \epsilon}{18 G_0} \right)^{1/3}$$

Please, note that the transition from an "area law" to an "area × radius law" is a very striking consequence of the simple assumption of allowing for scale dependent couplings!!

Numerical Results C



Numerical Results D



Message

1- A possible scale dependence of the gravitational coupling introduces an additional contribution to the stress energy tensor of the generalized field equations.

2- Integration constants plays a crucial role in this scale dependence approach!

3- The null energy condition allow us to get the Newton's function!!

4- Whereas for small black holes, the usual “area law” holds up to $\mathcal{O}(\epsilon)$, the opposite limit (which occurs when $G(r)$ deviates strongly from G_0) follows an “area × radius law”.