Dark Matter, Galaxy Rotation curves and Stationary axisymmetric solutions

Seminario en honor del Prof. Luis A. Herrera Cometta
... que 70 años no es nada...

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Outline

Motivation and Perspective

Motivation

The so-called *problem of the flattening of the galactic rotation curves,* one of the evidences of the existence of Dark Matter.

Perspective

Taking a look, with renovated interest and critical eye, at stationary and axisymmetric metrics:

- Carefully bringing out the hidden (legitimate and not-so-legitimate) assumptions
- Pursuing a better understanding of the relationship between geometry and observed quantities
- Above all, expressing the spacetime metric in terms of potentially observable quantities

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Dark Matter

What is Dark Matter (DM)?

- Matter we cannot see directly (non-luminous).
- Detected only through its gravitational effects.
- Most of the matter in the Universe? There are strong evidences of its existence (gravitational lensing, cosmological arguments...)

What is DM made of?

- **Baryonic DM:** Massive Compact Halo Objects (MACHOs): Brown dwarfs and Jupiter-sized planets, Cold stellar remnants, Cold Hydrogen (Elmegreen, Science **316** (2007) 1132)
- **Non-baryonic DM:**
  - Hot DM: particles moving at speeds close to c (massive neutrinos).
  - Warm DM: particles moving relativistically of m $\sim$ 1eV (gravitinos and photinos).
  - Cold Dark Matter: Weakly Interacting Massive Particles (WIMPs), predicted by some theoretical particle models (exotic DM).

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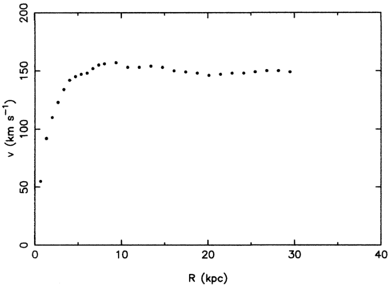
How (and when) it all begun: 1st observational facts

- Oort (1932). The velocity of stars (Doppler shift) in the local galactic neighborhood were *higher* than the escape velocity, thus there must be unseen matter. $M/L \sim 3$ (M: total amount of mass needed, L total amount of visible mass in terms of the luminosity)
- Zwicky (1933). Observations of the Coma Berenice Cluster of galaxies: some galaxies had velocities of about 1000 km/sec (greater than the escape velocity calculated using the Virial theorem). $M/L \sim 400 - 500$
- Rubin (late 1960s - early 1970s). Spiral galaxies are such that the rotation speed rises steadily from the center to the inner disk, and then becomes roughly constant (flat) in the outer parts. $M/L \sim 10$.

Flattening of the rotation curves



Rotation curve for the spiral galaxy NGC3198

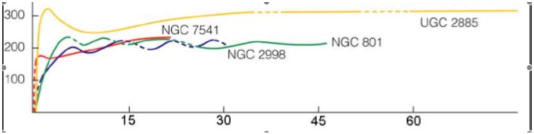
- Newtonian gravity

$$m\frac{V^2}{r} = m\frac{GM(r)}{r^2} \Rightarrow V = \sqrt{\frac{GM(r)}{r}} \Leftrightarrow M(r) = \frac{rV^2}{G}$$

where $M(r)$ is the total mass enclosed in a sphere of radius r.

Interpretation

- Almost all of the stars in a galaxy are concentrated in $r \lesssim 10$ kpc.
- According to $V = \sqrt{GM(r)/r}$, the rotation speed should rise to a maximum in the inner parts, and then fall as $V \propto 1/r^{1/2}$ outside a radius of ~ 10 kpc (Kepler). Clearly, this is not what happens!



Empirical modeling of μ and V for the galactic halo

- The most widely adopted model is the NFW Halo model (Navarro, Frenk and White, Ap J 490 (1997) 493):

$$\mu(r) = \frac{\mu_0}{(r/r_c)\,(1 + (r/r_c))^2}$$

- The resulting velocity is

$$V = V_0 \sqrt{\frac{r_c}{r} \ln\left(1 + \frac{r}{r_c}\right) - \frac{1}{1 + \frac{r}{r_c}}}$$

- Thus, the DM halo is assumed to be spherically symmetric.

Where we come from and where we want to go to

- Thus far, the problem of the galaxy rotation curves (RC) was dealt with in a Newtonian manner (besides assuming sphericall symmetry for the DM halo).

  The velocity of the stars and their distances to the galactic center are measured using standard astronomical methods (Doppler shift and astrometry). These are all the data we have.

- Next, we aim at treating the problem relativistically, in the hope that the amount of DM needed to explain the RC may be reduced (or not).

  After all, if we have a correct theory for gravitation, namely GR, why not use it?!

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Where we come from ...

- In order to do that we first need a suitable model of galaxy in General Relativity.
- Next, and in order to test the predictions of the theory against the observational data we have, we need to understand:
  - What is, in the GR context, the velocity V that astronomers measure.
  - What is, in the GR context, the distance from the galactic center that astronomers measure.
  - Further, and since measurements are made by observers (astronomers), we need to understand what is an observer and who are the observers carrying out the above measurements, i.e.: how is an astronomer to be represented within this context.

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Basic assumptions

Matter content

- Matter is modeled as a pressureless (dust) perfect fluid

$$T_{ab} = \mu u_a u_b, \quad u^a u_a = -1$$

where u^a is the four-velocity of the matter and μ its density.

Further, $\dot{u}^a = 0, \; \Theta = 0$, i.e.: the fluid is geodesic and non-expanding as a consequence of the contracted Bianchi identities.

Spacetime Geometry

- The spacetime is assumed to be stationary and axisymmetric, and the orthogonal transitivity condition holds. Coordinates $x^a = \rho, z, \phi, t$ *(equivalent to Weyl coordinates)* may be chosen s.t. the 2 KVs are $\vec{\xi} = \partial_\phi, \; \vec{\eta} = \partial_t$ and the metric is (see Bardeen, ApJ **162** (1970) 71)

$$ds^2 = e^{2\mu}\left(d\rho^2 + dz^2\right) + e^{2\psi}\left(d\phi - \omega dt\right)^2 - e^{2\nu} dt^2$$

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First consequences

- Due to EFEs and the assumed form of the metric, it follows that

$$\vec{u} = N\left(\Omega\partial_\phi + \partial_t\right), \quad \Omega = \Omega(\rho, z)$$

and N is a normalization factor. Recall that $\nabla_b T^{ab} = 0$ implies that the fluid must be geodesic and non-expanding: $\dot{u}_a = 0, \quad \Theta = 0$.

- The function Ω represents the angular velocity of the fluid, since

$$u^a = \frac{dx^a}{d\tau} \Rightarrow \frac{d\phi}{dt} = \Omega$$

For $\Omega = \mathrm{const}$ the fluid is said to be in rigid rotation, then one can always set $\Omega = 0$ without loss of generality by using a trivial coordinate change. Otherwise, it is said to be differentially rotating.

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Observers in axisymmetric and stationary spacetimes

- A family of observers in GR is described by a (f.d., timelike unit) vector field $\vec{v}$. Its integral curves are world lines of particles moving with that velocity.

- In an axisymmetric spacetime, a ZAMO *(Zero Angular Momentum Observer)* is defined to be an observer $\vec{v}$ whose angular momentum vanishes: $L = v_a \xi^a = 0$ where $\vec{\xi}$ is the axial KV.

  In the coordinates above: $\vec{v}$ $s.t.$ $v_\phi = 0$ which does not imply $v^\phi = 0$: (dragging of inertial systems).

$$\frac{d\phi}{ds} \neq 0 \Leftrightarrow \frac{d\phi}{dt} = \frac{d\phi}{ds}\frac{dt}{ds} \neq 0$$

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Who is the Astronomer?

- In the case of stationary and axisymmetric systems, one special ZAMO is that for which ρ and z are constant, which is precisely s.t. $v_a \propto t_{,a}$ i.e.: its 4-velocity is orthogonal to the hypersurfaces (slices) $t = \mathrm{const}$. This observer is at rest w.r.t. the frame of distant stars and is identified with an observer making astronomical observations. We call it Astronomer.

  For the metric

$$ds^2 = e^{2\mu}\left(d\rho^2 + dz^2\right) + e^{2\psi}\left(d\phi - \omega dt\right)^2 - e^{2\nu} dt^2$$

$$\vec{v} = N'\left(\omega\partial_\phi + \partial_t\right)$$

  and $\omega = d\phi/dt$ is the angular velocity of that observer.

What is V?

- Given a timelike unit vector field $\vec{u}$ such as the velocity field of the matter $\vec{u} = N(\Omega\partial_\phi + \partial_t)$ it may be decomposed with respect to the Astronomer in a standard way

$$\vec{u} = \gamma\left(\vec{v} + V\hat{e}_\phi\right), \qquad \gamma = \left(1 - V^2\right)^{-1/2}$$

where $\hat{e}_\phi = (g_{\phi\phi})^{-1/2}\partial_\phi$, and V is then the velocity with respect to the Astronomer.

In the coordinates $x^a = \rho, z, \phi, t$ above

$$V = e^{\psi-\nu}\left(\Omega - \omega\right) \qquad (1)$$

Note that V is proportional to the difference of angular velocities: the angular velocity of $\vec{u}$ and that of the ZAMO $\vec{v}$, which -in a sense- may be regarded as the angular velocity of spacetime itself.

What is the distance?

- Thus, for stationary and axisymmetric systems such as galaxies whose matter content is dust of four-velocity $\vec{u}$, it follows that $V = e^{\psi-\nu}(\Omega - \omega)$ gives the velocity of the fluid as measured by an observer at rest with respect to the frame of fixed stars.
- Clearly, the length of an orbit of the axial KV $\vec{\xi} = \partial_\phi$ in such spacetimes is $\ell = 2\pi\sqrt{g_{\phi\phi}}$, or, in the above coordinates $\ell = 2\pi e^{\psi}$, thus e^{ψ} is the proper circumferential radius, which in asymptotically flat spacetimes and at great distances from the axis $e^{\psi} \sim \rho$.

What we now know

- We have a relativistic description of a galaxy; namely: matter is dust $T_{ab} = \mu u_a u_b$, the spacetime is axisymmetric and stationary:

  $$ds^2 = e^{2\mu}\left(d\rho^2 + dz^2\right) + e^{2\psi}\left(d\phi - \omega dt\right)^2 - e^{2\nu} dt^2.$$

  The velocity of matter is $\vec{u} = N(\Omega\partial_\phi + \partial_t)$.

- The astronomer is $\vec{v} = N'(\omega\partial_\phi + \partial_t)$ or else $v_a = N'(0, 0, 0, -1)$.

- The velocity of the stars in the galaxy, as measured by the astronomer is $V = e^{\psi-\nu}\left(\Omega - \omega\right)$.

- The proper circumferential radius (which gives an idea of the distance to the galactic centre) is $D = e^{\psi}$.

What we have assumed

- The material content is dust. Quite legitimate, but we are discarding, for instance, light (of which galaxies have a lot), neutrinos, etc., or the fact that most galaxies seem to have a BH at its centre.
- The geometry is axially symmetric and stationary, and satisfies the orthogonal transitivity condition. Quite legitimate, but we should be aware that the latter is a rather strong geometric condition, basically used for the sake of simplicity, but that forbids -for instance- convective motions.
- The distance of a given star to the centre of the galaxy, as measured by the astronomer, is given by ρ. Not sure about how good an approximation this is... In any case, this assumption is always hidden in the literature.

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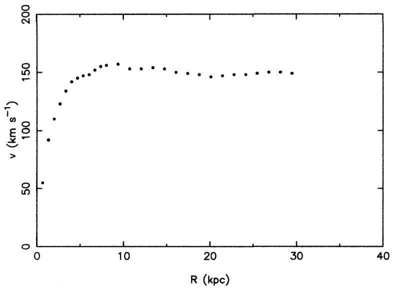
What we would like to do...

- Clearly, we'd like to find an exact solution to the EFEs that fits the curve $V = V(Distance)$, and then calculate what is the density μ it corresponds to.



But...

... and the obstacles we face

- Finding explicit exact solutions to the above problem is extremely difficult.
- It would be extremely interesting to look at this problem within the framework of the exact solutions for rotating disks (rigidly and differentially) of dust found by Neugebauer, Meinel, Ansorg and collaborators (see especially: GRG (2000) **32** 1365, Phys. Lett. A (1997) **210** 160, and references cited therein.)
- The curve $V = V(Distance)$ is only known for stars in the equatorial plane of the galaxy.
- One would like a model in which a BH existed at the centre (as this is what is suspected to happen in most galaxies). It would be interesting to explore the scenario dealt with in Ansorg and Hennig (CQG (2008) **25** 22200, and references cited therein)

- The coordinate ρ is not, in general, the distance to the galactic centre as measured by astronomers (it only is at large distances and assuming asymptotic flatness).

- It would be nice and convenient to have a good translation into GR of the distance an astronomer estimates between the centre of a distant galaxy and a star located in the equatorial plane. Note: all plots of V are given as $V = V(\rho)$ assuming ρ is the distance to the galactic centre.

- The observer we called *the Astronomer* is not, in general, geodesic. It only is at large spatial distances.

What has been done

There have been, roughly, three kinds of attempts to explain relativistically the RC in galaxies (aside from PPN approximations).

1. Modify GR theory. (Moffat, Carmeli ...)
2. Assume from the start an spherically symmetric DM halo, modeling the matter as a scalar field.
   Of course you can get excellent fits for the RC!
   Notice though that it is extremely difficult (if possible) to match an axisymmetric solution to a spherically symmetric one.
3. Using GR and assuming baryonic matter modeled as dust, just as we've been describing so far.
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The attempts in GR

Both CT and BG assume the galaxy to be rigidly rotating; i.e.: $\vec{u} \propto \partial_t$ or $\Omega = 0$, the metric and consequently the field equations are then much simpler.

$$ds^2 = -(dt - Nd\phi)^2 + \rho^2 d\phi^2 + e^{2\mu}\left(d\rho^2 + dz^2\right)$$

CT Mathematically, it is very weak (to say the least!) Physically, it contains a surface layer in the equatorial plane of negative mass density... its merit, though is that it was the first time that GR was used to address this problem.
It got many criticisms (Korzynski, Bonnor, Cross, Zingg et al., Vogt and Letelier,...)

BG They produced a Toy Model in which

$$V(0,z) \quad \text{is regular only for} \quad |z| \leq 1kpc$$

and other functions appearing in the solution are only regular in the region of interest (but not beyond that), but the model is exactly solvable.

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The BG model

$$V = \frac{N}{\rho}, \quad \mu \approx \frac{\beta}{\rho}\left(N_\rho^2 + N_z^2\right), \quad \beta \text{ constant}$$

$$V(\rho, 0) = \frac{V_0}{\rho}\left(R - r_0 + \sqrt{r_0^2 + \rho^2} - \sqrt{R^2 + \rho^2}\right)$$

$$\mu(\rho \to 0) \sim \beta \frac{V_0^2}{r_0^2}, \quad \mu(r_0 << \rho << R) \sim \frac{1}{r^2}, \quad \mu(\rho \geq R) \sim \frac{1}{r^6}$$

- The chosen values are $r_0 \sim 1$ kpc (bulge radius), $R \sim 100$ kpc, $V_0 \sim 200$ km/sec (velocity in the flat regime).
- Integrating μ it gives a total mass $M \sim 10^{11} M_\odot$.

Reduction of density in BG model

Recall that the Newtonian estimation of the mass density in terms of the velocity V was

$$\mu_N = \frac{1}{4\pi G}\frac{V^2 + 2rVV'}{r^2}$$

Taking μ in BG and comparing with the above

$$\frac{\mu}{\mu_N} = \beta\left(1 + \frac{r^2}{V^2 + 2rVV'}\right)$$

Now, the constant β has to be chosen globally, choosing it such that in the linear regime ($\sim$ within the bulge) both densities coincide, we get, for the flat regime

$$\frac{\mu}{\mu_N}\Big|_{\text{linear}} = 1 = \beta\frac{4}{3} \Rightarrow \frac{\mu}{\mu_N}\Big|_{\text{flat}} = \beta = \frac{3}{4}$$

Thus, the amount of DM needed has been reduced by 30%.

Our contribution

We do not think that we can do entirely away with DM (there are evidences of its existence), but certainly the amount needed to explain RC can be reduced by using full GR, what is more: since DM seems to be made up of hot, warm and cold particles (this last one being exotic particles, not yet detected), a reduction in the total amount may result in a significant reduction in the proportion of one of the above spices.

Hence, the need for a new look at stationary and axisymmetric solutions, having in mind all the problems we mentioned.

- Writing the metric in terms of potentially observable quantities, so that observational data could be fed in from the start.
- Setting up new coordinate systems with a clearer geometric meaning, and well suited to numerical calculations.

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Metric in terms of potentially observable quantities.

- Idea: Treat $V(\rho, z)$ and $\Omega(\rho, z)$ as data (inputs in the problem). Recall $\vec{u} = N(\Omega\partial_\phi + \partial_t)$.
- We choose coordinates $x^a = \rho, z, \phi, t$ as previously, but write instead the metric as:

$$g_{ab} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & n \\ 0 & 0 & n & c \end{pmatrix}, \qquad bc - n^2 = -\rho^2$$

- With these assumptions

$$V = \frac{1}{\rho}(b\Omega + n), \qquad u^a = \frac{b^{1/2}}{\rho\sqrt{1 - V^2}}(0, 0, \Omega, 1)$$

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Exploiting the geodesic condition

Recall that $\dot{u}^a = 0$ as a result of the contracted Bianchi identities.

$$\dot{u}^a = 0 \quad \Rightarrow \quad \Gamma^a_{bc} u^b u^c = 0$$

which boils down to

$$\begin{aligned} b_z \Omega^2 + 2 n_z \Omega + c_z &= 0 \\ b_\rho \Omega^2 + 2 n_\rho \Omega + c_\rho &= 0 \end{aligned}$$

Taking into account $bc - n^2 = -\rho^2$ and defining $B \equiv \ln b$ and $\Phi \equiv (\rho + \rho V^2)/2V$ the above system is

$$\begin{aligned} n_z &= B_z (n - \Phi) \\ n_\rho &= B_\rho (n - \Phi) + \frac{1}{V} \end{aligned}$$

Exploiting the geodesic condition

Recall that $\dot{u}^a = 0$ as a result of the contracted Bianchi identities.

$$\dot{u}^a = 0 \quad \Rightarrow \quad \Gamma^a_{bc} u^b u^c = 0$$

which boils down to

$$\begin{aligned} b_z \Omega^2 + 2 n_z \Omega + c_z &= 0 \\ b_\rho \Omega^2 + 2 n_\rho \Omega + c_\rho &= 0 \end{aligned}$$

Taking into account $bc - n^2 = -\rho^2$ and defining $B \equiv \ln b$ and $\Phi \equiv (\rho + \rho V^2)/2V$ the above system is

$$\begin{aligned} n_z &= B_z (n - \Phi) \\ n_\rho &= B_\rho (n - \Phi) + \frac{1}{V} \end{aligned}$$

...

Demanding $n_{\rho z} = n_{z\rho}$ and $B_{\rho z} = B_{z\rho}$, one finally gets

$$b = \frac{1 - V^2}{V^2} \frac{1}{H'(\rho V)}, \quad c = \frac{n^2 - \rho^2}{b}$$

$$n = \frac{1}{2} \left(\frac{1 - V^2}{V^2} \frac{H(\rho V)}{H'(\rho V)} + \rho \frac{1 + V^2}{V^2} \right)$$

$$\Omega = -\frac{1}{2} \left(H(\rho V) + \rho V \, H'(\rho V) \right) \equiv -\frac{1}{2} \left(\rho V \, H(\rho V) \right)'$$

where $H(\rho V)$ is an arbitrary function of its argument, and a prime indicates differentiation w.r.t. this argument.

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The metric in terms of potentially observable quantities

Thus,

$$g_{\phi\phi} = \frac{1 - V^2}{V^2} \frac{1}{H'(\rho V)},$$

$$g_{\phi t} = \frac{1}{2} \left(\frac{1 - V^2}{V^2} \frac{H(\rho V)}{H'(\rho V)} + \rho \frac{1 + V^2}{V^2} \right),$$

$$g_{tt} = \frac{g_{\phi t}^2 - \rho^2}{g_{\phi\phi}}$$

And also,

$$\Omega = -\frac{1}{2} \left(\rho V \, H(\rho V) \right)'$$

Therefore, the metric potentials have been expressed in terms of V (the rotation velocity – observable), and $H(\rho V)$ an arbitrary function of its argument that, if Ω (the angular velocity of the fluid – observable) is given, can be integrated out.
Recall that $g_{zz} = g_{\rho\rho}$ are fixed by the field equations.

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What is left:

1. Write down EFEs in terms of V and $H(\rho V)$ (and/or Ω).
2. Derive the energy density μ from there.
3. Compare μ with μ_L, the density of the luminous matter.
4. Draw conclusions...